On a Class of Adaptive Suboptimal Riccati–based Controllers

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Abstract

In this paper a nonlinear modification is given to the standard **control law** resulting from a linear optimal control problem formulated in **state–space**. This law weights the value of the state feedback matrix with a gain factor whose derivative is proportional to the difference of a weighted inner product of the state vector and the logarithm of the gain factor. The result is completely general and therefore applies to **multi–input multi–output systems**. An example illustrates the extent to which the convergence rate is increased using the nonlinear feedback law for large values of the state.

Introduction

This paper is concerned with modifying the control laws that are derived from the infinite horizon optimal control problem for linear systems. To reiterate the well-known result [1, 2], suppose that we have the standard linear time-invariant system with

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx, \end{aligned} \tag{1}$$

where $A \in \mathbf{R}^{n,n}$, $B \in \mathbf{R}^{n,m}$, and $C \in \mathbf{R}^{n,p}$. Suppose, too, that the system is both observable and reachable. The infinite horizon optimal control problem is formulated as the minimization of the cost functional $J(u)$ over all possible inputs u where

$$J(u) = \int_0^\infty y^T y + u^T u \; dt. \tag{2}$$

The minimizing solution may be written as $u = -B^T P x$ where P is the unique positive definite stabilizing solution to the algebraic Riccati equation

$$A^T P + PA - PBB^T P + C^T C = 0. \tag{3}$$

Less well known, perhaps, is that this Riccati solution provides a family of stabilizing non-optimal solutions parameterized over a scalar. In other words, the feedback $u = -\frac{1}{2}(1+\delta)B^T Px$ is also stabilizing where $\delta > 0$ [5]. The aim of this paper is to show that there exists an adaptive law for δ which produces a stabilizing solution as well. This adaptive law enables to controller to operate in two regimes: one with slower dynamics for states with small magnitudes and one with faster dynamics for states with large magnitudes. An application for such a scheme could be in target acquisition and tracking.

Non-Linear Controller

Suppose we augment the control law $u = -\kappa(1+\delta)B^T Px$ where the gain factor δ is allowed to vary according to the equation

$$\dot{\delta} = -\alpha \log(1+\delta) + \frac{\beta}{\log(1+\delta)+1} x^T \left((2\kappa(1+\delta)-1)\, PBB^T P + C^T C\right) x \tag{4}$$

for some positive constants α and β with $\kappa > \frac{1}{2}$. This control law is reminiscent of the square-law adaptive control algorithm of Willems and Byrnes [7], with the exception that there is a relaxation term whose derivative is proportional to its logarithm. Note that the square-law is applied here to a vector-valued argument as opposed to a scalar-valued argument. The logarithmic relaxation term was inspired by a result by Ilchman and Townley [3] in which a discrete-time version of the Willems-Byrnes nonlinear controller had time-step sizes that varied logarithmically.

Immediately one property of the dynamics of these equations is evident, which is given in the following lemma.

Lemma 1 *Suppose the ordinary differential equation in (4) has the initial condition $\delta(t_0) = \delta_0 \geq 0$, then $\delta \geq 0 \;\forall\, t > t_0$.*

Proof. The proof is given by noting that $\dot{\delta} = -\alpha \log(1+\delta)$ is a first order differential equation with $\dot{\delta} = 0$ when $\delta = 0$ and that the nonhomogeneous term
$\beta(\log(1+\delta)+1)^{-1}\, x^T((2\kappa(1+\delta)-1)\, PBB^T P + C^T C)x \geq 0 \;\forall\, \delta \geq 0,\; x \in \mathcal{R}^n$. □

It is also worth noting that $\delta = 0$ is a stable fixed point in the homogeneous differential equation $\dot{\delta} = -\alpha \log(1+\delta)$. This may be shown by observing that the equation above may be approximated by the linear equation $\dot{\delta} = -\alpha\delta$ in a small neighborhood about $\delta = 0$, $x = 0$.

Next, let us define the function

$$V(x,\delta) = (1+\beta)x^T Px + (1+\delta)\log(1+\delta). \tag{5}$$

Differentiating this equation we have

$$\begin{aligned}
\dot{V}(x,\delta) &= -(1+\beta)x^T((2\kappa(1+\delta)-1)\, PBB^T P + C^T C)x + \dot{\delta}(\log(1+\delta)+1) \\
&= -(1+\beta)x^T((2\kappa(1+\delta)-1)\, PBB^T P + C^T C)x - \alpha\log(1+\delta)(\log(1+\delta)+1) \\
&\qquad + \beta x^T((2\kappa(1+\delta)-1)\, PBB^T P + C^T C)x(\log(1+\delta)+1)^{-1}(\log(1+\delta)+1) \\
&= -x^T((2\kappa(1+\delta)-1)\, PBB^T P + C^T C)x - \alpha\log(1+\delta)(\log(1+\delta)+1).
\end{aligned} \tag{6}$$

We are now prepared to give the main result:

Theorem 2 *Let $A \in \mathbf{R}^{n,n}$, $B \in \mathbf{R}^{n,m}$, $C \in \mathbf{R}^{n,p}$, with $[A,B]$ reachable and $[A,C]$ observable. Also, let the constants $\alpha > 0$, $\beta > 0$ and $\kappa > \frac{1}{2}$. Then*
i) there exists a positive definite solution P to (3), such that

$$\dot{x} = (A - \kappa BB^T P)x \tag{7}$$

is stable and
ii) the ordinary differential equation given by the system

$$\begin{aligned} \dot{x} &= (A - \kappa(1+\delta)BB^T P)x & , \quad x(t_0) \in \mathcal{R}^n \\ \dot{\delta} &= -\alpha \log(1+\delta) + \frac{\beta}{\log(1+\delta)+1} x^T((2\kappa(1+\delta)-1)\, PBB^T P + C^T C)x & , \quad \delta(t_0) \geq 0 \end{aligned} \tag{8}$$

is asymptotically stable with $\lim_{t\to\infty} \delta(t) = 0$ and $\lim_{t\to\infty} x(t) = 0$.

Proof. Statement *i)* is a standard result from optimal control theory [1, 2, 5]. To prove *ii)*, we start by letting $\mathbf{D} = \mathcal{R}^n \times \mathcal{R}^+$ where $\mathcal{R}^+$ represents the nonnegative real numbers. Lemma 1 ensures that $[x, \delta]$ remain in $\mathbf{D}$ $\forall\, t > t_0$. From (5) $V(x,\delta) > 0 \;\forall\, [x,\delta] \in \mathbf{D}$. Let $\mathbf{M} = \mathcal{N}((2\kappa - 1)PBB^T P + C^T C) \times \{0\}$, where $\mathcal{N}$ is the null space operator. From (6) $\dot{V}(x,\delta) < 0 \;\forall\, [x,\delta] \in \mathbf{D} - \mathbf{M}$ and $\dot{V}(x,\delta) = 0 \;\forall\, [x,\delta] \in \mathbf{M}$. Since the Lyapunov function satisfies the conditions above and is continuously differentiable over $\mathbf{D}$, and by Lemma 1 the solution $[x,\delta]$ remains in $\mathbf{D}$ $\forall\, t > t_0$, then by LaSalle's Theorem [4, 6] x must converge to the largest invariant set in $\mathbf{M}$. We note, however, that $[x,\delta] \in \mathbf{M} \;\Rightarrow\; \delta = 0$, and therefore that the solution to (8) on $\mathbf{M}$ is defined by a *linear* differential equation (7). Since (7) is stable for any $x_0 \in \mathcal{R}^n$ by *i)*, the largest invariant set in $\mathbf{M}$ is $[x,\delta] = [0,0]$, completing the proof. □

Numerical Example

A numerical example is given here to illustrate the dynamics of the nonlinear controller derived in the previous section. Suppose the system mentioned in (8) takes scalar values with $A = 0.05$, $B = 1$, $C = 0.1$, $\alpha = 7$, $\beta = 160$, $\kappa = 0.75$, $x_0 = 1$, and $\delta_0 = 0$. The positive definite solution to the Riccati equation in (3) is $P = 0.1618$. The results of the simulation are given in Figures 1 and 2. Those results that are labeled "Linear" are for simulations in which the value of δ remains fixed at $\delta = 0$. Those labeled "Nonlinear" allow the value of δ to vary according to the update laws above. Note that for large values of the state x, the convergence rate of the nonlinear system is much faster as evidenced by the value of δ. As the state approaches zero, the value of $\delta \approx 0$, and therefore the convergence rate is predominantly governed by the linear dynamics of (7).

Conclusions

In this paper a modified nonlinear control law is given which is based on a linear control law derived from an algebraic Riccati equation. The impetus for studying this system came from considering a Kalman filter with slow dynamics in the presence of occasional step-function disturbances. In such a case, the convergence to steady-state is too slow to effectively reject the occasional disturbance, while the option of speeding up the dynamics obviates the optimality of the filter.

Acknowledgement

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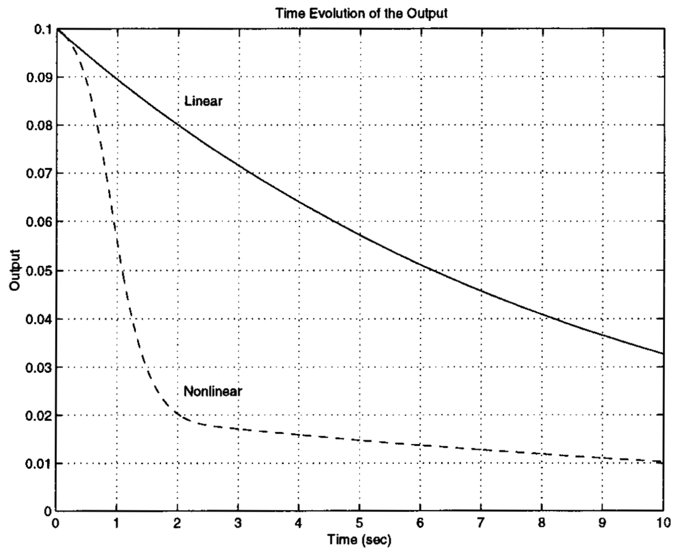

Figure 1: Convergence Comparison

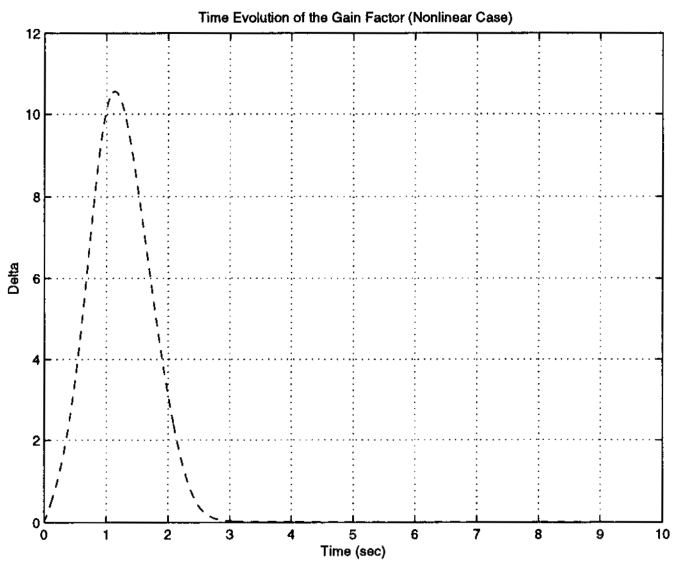

Figure 2: Gain Factor Evolution